TRANSFINITE ORDINALS by Jacques Bailhache, January-march 2018

Any ordinal can be defined as the least ordinal strictly greater than all ordinals of a set : the empty set for 0, $\{\alpha\}$ for the successor of α, $\{\alpha_0, \alpha_1, \alpha_2, ...\}$ for an ordinal with fundamental sequence $\alpha_0, \alpha_1, \alpha_2, ...$

1 Algebraic notation

We define the following operations on ordinals :

- addition : $\alpha + 0 = \alpha; \alpha + suc(\beta) = suc(\alpha + \beta); \alpha + lim(f) = lim(n \mapsto \alpha + f(n))$
- multiplication : $\alpha \times 0 = 0; \alpha \times suc(\beta) = (\alpha \times \beta) + \alpha; \alpha \times lim(f) = lim(n \mapsto \alpha \times f(n))$
- exponentiation : $\alpha^0 = 1; \alpha^{suc(\beta)} = \alpha^\beta \times \alpha; \alpha^{lim(f)} = lim(n \mapsto \alpha^{f(n)})$

2 Veblen functions

$\varepsilon_0 = lim\ \omega, \omega^\omega, \omega^{\omega^\omega}, \ldots; \varepsilon_1 = lim\ \varepsilon_0, {\varepsilon_0}^{\varepsilon_0}, {\varepsilon_0}^{{\varepsilon_0}^{\varepsilon_0}}, \ldots = lim\ \varepsilon_0 + 1, \omega^{\varepsilon_0+1}, \omega^{\omega^{\varepsilon_0+1}}, \ldots; \zeta_0 = lim\ 0, \varepsilon_0, \varepsilon_{\varepsilon_0}, \ldots$

$\omega^\alpha = \varphi_0(\alpha) = \varphi(0, \alpha); \varepsilon_\alpha = \varphi_1(\alpha) = \varphi(1, \alpha); \zeta_\alpha = \varphi_2(\alpha) = \varphi(2, \alpha)$

$\varphi(\ldots, \beta, 0, \ldots, 0, \gamma)$ is the $(1+\gamma)^{th}$ common fixed point of the functions $\xi \mapsto \varphi(\ldots, \delta, \xi, 0, \ldots, 0)$ for all $\delta < \beta$.

$\varphi(\alpha_n, \ldots, \alpha_0, \beta)$ may also be written $\varphi_{\alpha_n, \ldots, \alpha_0}(\beta)$ or $\varphi_{\Omega^n \times \alpha_n + \ldots + \alpha_0}(\beta)$ or $\varphi(\Omega^n \times \alpha_n + \ldots + \alpha_0, \beta)$ or $\begin{pmatrix} \beta & \alpha_0 & \ldots & \alpha_n \\ 0 & 1 & \ldots & n+1 \end{pmatrix}$

3 Simmons notation

$Fixfz = f^w(z+1) =$ least fixed point of f strictly greater than z ; $Next = Fix(\alpha \mapsto \omega^\alpha)$

$[0]h = Fix(\alpha \mapsto h^\alpha \omega)$; $[1]hg = Fix(\alpha \mapsto h^\alpha g \omega)$; $[2]hgf = Fix(\alpha \mapsto h^\alpha g f \omega)$; etc...

Correspondence with Veblen's φ : $\varphi(1+\beta, \alpha) = ([0]^\beta Next)^{1+\alpha}\omega$

If $\gamma > 0, \varphi(\gamma, \beta, \alpha) = \varphi(\gamma \times \Omega + \beta, \alpha) = ([0]^{\gamma \times \Omega + \beta} Next)^{1+\alpha}\omega = ([0]^\beta(([0]^\Omega)^\gamma Next))^{1+\alpha}\omega = ([0]^\beta(([1][0])^\gamma Next))^{1+\alpha}\omega$

If $\delta > 0$ or $\gamma > 0, \varphi(\delta, \gamma, \beta, \alpha) = \varphi(\delta \times \Omega^2 + \gamma \times \Omega + \beta, \alpha) = ([0]^{\Omega^2 \times \delta + \Omega \times \gamma + \beta} Next)^{1+\alpha}\omega = ([0]^\beta([0]^\Omega)^\gamma([0]^{\Omega^2})^\delta Next)))^{1+\alpha}\omega = ([0]^\beta(([1][0])^\gamma(([1]^2[0])^\delta Next)))^{1+\alpha}\omega$, with $[0]^{\Omega^n} = [1]^n[0]$.

Rationalization of $\varphi : \varphi(1+\beta, \alpha) = \varphi'(\beta, 1+\alpha) => \varphi'(\beta, \alpha) = ([0]^\beta Next)^\alpha \omega; \varphi(\gamma, \beta, \alpha) = \varphi'(\gamma, \beta, 1+\alpha)$

4 RHS0 notation

We start from 0, if we don(t see any regularity we take the successor, if we see a regularity, if we have a notation for this regularity, we use it, else we invent it, then we jump to the limit.

$Hfx = lim\ x, fx, f(fx), \ldots; R_1 fgx = lim\ gx, fgx, ffgx, \ldots; R_2 fghx = lim\ hx, fghx, fgfghx, \ldots$

Correspondence with Simmons notation : $\ldots, [3] \to R_5, [2] \to R_4, [1] \to R_3, [0] \to R_2, Next \to R_1, \omega \to Hsuc\ 0$

5 Tree ordinals

A tree ordinal a belongs to the tree ordinal class Ω_n($n \in \mathbb{N}$ if either a = 0, a = a' + 1 for some tree ordinal a' belonging to the tree ordinal class Ω_n, or a is a function from Ω_k to Ω_n for some k ¡ n.

To any tree ordinal a, we can associate a corresponding ordinal $\alpha = |a|$ obtained by ignoring the choice of particular fundamental sequences, and defined by : $|0| = 0$; $|a+1| = |a| + 1$; $|a| = sup|a[b]|$ if a is a function from Ω_k to Ω_n.

We can define the following extension of the Fast Growing Hierarchy (which corresponds to the case n=0) :

- $F_n(0, b) = b + 1$
- $F_n(a+1, b) = [F_n(a, \bullet)]^b(b)$
- $(F_n(a, b))[c] = F_n(a[c], b)$ if a is a function from Ω_k to Ω_{n+1} with $k < n$
- $(F_n(a, b)) = F_n(a[b], b)$ if a is a function from Ω_n to Ω_{n+1}

6 Ordinal collapsing functions

These functions use uncountable ordinals to define countable ordinals.

We define sets of ordinals that can be built from given ordinals and operations, then we take the least ordinal which is not in this set, or the least ordinal which is greater than all contable ordinals of this set.

These functions are extensions of functions on countable ordinals, whose fixed points can be reached by applying them to an uncountable ordinal, for example :

- Buchholz ψ_0 : $\psi_0(\alpha) = \omega^\alpha$ if $\alpha < \varepsilon_0; \psi_0(\Omega) = \varepsilon_0$ which is the least fixed point of $\alpha \mapsto \omega^\alpha$.
- Madore's ψ : $\psi(\alpha) = \varepsilon_\alpha$ if $\alpha < \zeta_0; \psi(\Omega) = \zeta_0$ which is the least fixed point of $\alpha \mapsto \varepsilon_\alpha$.
- Feferman's θ : $\theta(\alpha, \beta) = \varphi(\alpha, \beta)$ if $\alpha < \Gamma_0$ and $\beta < \Gamma_0; \theta(\Omega, 0) = \Gamma_0$ which is the least fixed point of $\alpha \mapsto \varphi(\alpha, 0)$.
- Taranovsky's C : $C(\alpha, \beta) = \beta + \omega^\alpha$ if α is countable; $C(\Omega_1, 0) = \varepsilon_0$ which is the least fixed point of $\alpha \mapsto \omega^\alpha$.

Some general formulas for ordinal collapsing functions are :

- $\psi_\nu(0) = z(\nu)$ (for example : $\psi_\nu(0) = \Omega_\nu$, or $\psi_0(0) = 1; \psi_{1+\nu}(0) = \Omega_{1+\nu} = \omega_{1+\nu}$
- $\psi_\nu(suc\ \alpha) = f(\psi_\nu(\alpha))$
- $\psi_\nu(lim\ h) = lim(\psi_\nu \circ h)$ (with $lim = Lim_0$)
- $\psi_\nu(Lim_{\kappa+1} h) = Lim_{\kappa+1}(\psi_\nu \circ h)$ if $\kappa < \nu$, or with fundamental sequence notation : $\psi_\nu(\alpha)[\eta] = \psi_\nu(\alpha[\eta])$
- $\psi_\nu(Lim_{\kappa+1} h) = lim[\psi_\nu(h((\psi_\kappa \circ h)^\bullet(\zeta)))]$ if $\kappa \geq \nu$, with $\zeta = 0$ or 1 or $\psi_\kappa(0)$ for example.

Name	Symbol	Algebraic	Veblen	Simmons	RHS0	Madore	Taranovsky
Zero	0	0			0		0
One	1	1	$\varphi(0,0)$		suc 0		C(0,0)
Two	2	2			suc (suc 0)		C(0,C(0,0))
Omega	ω	ω	$\varphi(0,1)$	ω	H suc 0		C(1,0)
		$\omega+1$			suc (H suc 0)		C(0,C(1,0))
		$\omega\times 2$			H suc (H suc 0)		C(1,C(1,0))
		ω^2	$\varphi(0,2)$		H (H suc) 0		C(C(0,C(0,0)),0)
		ω^ω	$\varphi(0,\omega)$		H H suc 0		C(C(1,0),0)
Epsilon zero	ε_0	ε_0	$\varphi(1,0)$	$Next\ \omega$	$R_1 H suc\ 0$	$\psi(0)$	$C(\Omega_1,0)$
		ε_1	$\varphi(1,1)$	$Next^2\omega$	$R_1(R_1 H) suc\ 0$	$\psi(1)$	$C(\Omega_1,C(\Omega_1,0))$
		ε_ω	$\varphi(1,\omega)$	$Next^\omega\omega$	$HR_1 H suc\ 0$	$\psi(\omega)$	$C(C(0,\Omega_1),0)$
		$\varepsilon_{\varepsilon_0}$	$\varphi(1,\varphi(1,0))$	$Next^{Next\omega}\omega$	$R_1 H R_1 H suc\ 0$	$\psi(\psi(0))$	$C(C(C(\Omega_1,0),\Omega_1),0)$
Zeta zero	ζ_0	ζ_0	$\varphi(2,0)$	$[0] Next\ \omega$	$R_2 R_1 H suc\ 0$	$\psi(\Omega)$	$C(C(\Omega_1,\Omega_1),0)$
Eta zero	η_0	η_0	$\varphi(3,0)$	$[0]^2 Next\ \omega$	$R_2(R_2 R_1) H suc\ 0$		$C(C(\Omega,C(\Omega,\Omega)),0)$
			$\varphi(\omega,0)$	$[0]^\omega Next\ \omega$	$H R_2 R_1 H suc\ 0$		$C(C(C(0,\Omega_1),\Omega_1),0)$
Feferman -Schütte	Γ_0	Γ_0	$\varphi(1,0,0)$ $=\varphi(2\mapsto 1)$	$[1][0] Next\ \omega$	$R_3 R_2 R_1 H suc\ 0$ $= R_{3\ldots 1} H suc\ 0$	$\psi(\Omega^\Omega)$	$C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),0)$
Ackermann			$\varphi(1,0,0,0)$ $=\varphi(3\mapsto 1)$	$[1]^2[0] Next\ \omega$	$R_3(R_3 R_2) R_1 H suc\ 0$	$\psi(\Omega^{\Omega^2})$	
Small Veblen ordinal			$\varphi(\omega\mapsto 1)$	$[1]^\omega[0] Next\ \omega$	$H R_3 R_2 R_1 H suc\ 0$	$\psi(\Omega^{\Omega^\omega})$	$C(\Omega_1^\omega,0)$ $=C(C(C(C(0,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Large Veblen ordinal			least ord. not rep.	$[2][1][0] Next\ \omega$	$R_4 R_3 R_2 R_1 H suc\ 0$ $= R_{4\ldots 1} H suc\ 0$	$\psi(\Omega^{\Omega^\Omega})$	$C(\Omega_1^{\Omega_1},0)$ $=C(C(C(C(\Omega_1,\Omega_1),$ $\Omega_1),\Omega_1),0)$
Bachmann-Howard				least ord. not rep.	$R_{\omega\ldots 1} H suc\ 0$	$\psi(\varepsilon_{\Omega+1})$	$C(C(\Omega_2,\Omega_1),0)$